(a) Show that, for $0 \le x \le 1$,

$$\sum_{k=m}^{n} a_k x^k = \sum_{j=m}^{n-1} S_m^j (x^j - x^{j+1}) + S_m^n x^n.$$

(b) Show that

$$\lim_{x \to 1^-} \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k.$$

Hint: Estimate the left hand side of the formula in (b).

---

4. (30p) Let $\{f_j\}$ be a sequence of real-valued functions in $L^1(X, \mu)$ such that $f_j \to f$ a.e. with $f \in L^1(X, \mu)$. Suppose $\{g_j\}$ is a sequence of functions in $L^1(X, \mu)$ such that $|f_j| \le g_j$, $g_j \to g$ a.e. for some $g \in L^1(X, \mu)$, and also $g_j \to g$ in L^1. Prove that

$$\int_X f d\mu = \lim_{j \to \infty} \int_X f_j d\mu$$

---

5. Let $\mathcal{H}$ be a Hilbert space, $P \colon \mathcal{H} \to \mathcal{H}$ a bounded linear operator, $P^* \colon \mathcal{H} \to \mathcal{H}$ its adjoint, and $\mathcal{R}(P)$, $\mathcal{N}(P)$ its range and nullspace, respectively.

(a) (10p) Show that $\mathcal{N}(P^*) = \mathcal{R}(P)^\perp$ and $\overline{\mathcal{R}(P^*)} = \mathcal{N}(P)^\perp$.

(b) (15p) Show that $\mathcal{R}(P^*)$ is closed if $\mathcal{R}(P)$ is closed.

(c) (15p) Assume that $P^2 = P$. Show that the following are equivalent (you may use parts (a) and (b) even if you did not do them):

(i) $\mathcal{R}(P)$ is closed and $||x - Px|| = \inf_{y \in \mathcal{R}(P)} ||x - y||$ for every $x \in \mathcal{H}$.

(ii) $P = P^*$.

---

6. (30p) For each $N \in \mathbb{R}$, define in $\mathbb{R}^n$ the measures $d\mu_N := (1 + |x|)^N dx$ and, for each $1 \le p < \infty$, the norms

$$||f||_{p,N} := ||f||_{L^p(\mathbb{R}^n, d\mu_N)} = \left(\int_{\mathbb{R}^n} |f(x)|^p (1 + |x|)^N dx \right)^{1/p}.$$

(a) Show that for every $t > 0$, there is a constant C_t such that for every $1 \le r < p < \infty$ and every

$$N_t := \frac{Np + n(p - r) + t(p - r)}{r}$$

the estimate

$$||f||_{r,N} \le C_t ||f||_{p,N_t}$$

holds.