MATH 240: Real Analysis
Qualifying Exam. May 26, 2006

**General instructions**: 3 hours. No books or notes. Be sure to motivate all (nontrivial) claims and statements. You may use without proof any result proved in the text. You need to reprove any result given as an exercise.

**Specific instructions**: Do problems 1, 5, and 6. Choose *two* of the three problems 2, 3, and 4. Be sure to indicate which two of these you want graded.

**Notation**: $m = dx$ denotes the Lebesgue measure. $(X, \mathcal{M}, \mu)$ is a measure space.

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1. (50p) Determine if the statements below are **True** or **False**. If **True**, give a brief proof. If **False**, give a counterexample (or prove your assertion in another way, if you prefer). If you claim an assertion follows from a theorem in the text, name the theorem (or describe it otherwise) and explain carefully how the conclusion follows.

(a) If $f \in C([0,1])$, $f'$ exists a.e. $(m)$ and $f' = 0$ a.e. $(m)$, then $f$ is constant.

(b) If $X \supset E_1 \supset E_2 \supset \ldots$ are $\mathcal{M}$-measurable sets such that $\mu(\cap_j E_j) = 0$, then $\lim_{j\to\infty} \mu(E_j) = 0$.

(c) Suppose $\{f_j\}$ is a sequence in $L^1(X,\mu)$ with $f_1 \geq f_2 \geq \ldots \geq 0$ a.e. $(\mu)$, and let $f(x) = \lim_{j\to\infty} f_j(x)$ a.e. $(\mu)$. Then $\int_X f d\mu = \lim_{j\to\infty} \int_X f_j d\mu$.

(d) Let $\nu$ be a finite signed measure on $X$, and $|\nu|$ its total variation. Then, there is $f \in L^1(X, |\nu|)$ such that for every $g \in L^1(X,\nu)$, $\int_X g d\nu = \int_X g f d|\nu|$.

(e) Let $\mathcal{X}$ be a Banach space and $\mathcal{X}^*$ its dual. Let $\{x_j^*\}$ be a sequence in $\mathcal{X}^*$ such that $\lim_{j\to\infty} x_j^*(x)$ exists (as a complex number) for every $x \in X$. Then, there is $x^* \in X^*$ such that $x^*(x) = \lim_{j\to\infty} x_j^*(x)$ for every $x \in X$.

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2. (30p) Let $\{f_j\}$ be a sequence in $AC([0,1])$ such that that $\lim_{j\to\infty} f_j(0) = c$ and, for some $g \in L^1([0,1], dx)$, $f_j' \to g$ in $L^1$.

(a) Show that $f(x) = \lim_{j\to\infty} f_j(x)$ exists for all $x \in [0,1]$.

(b) Show that $f \in AC([0,1])$ and $f' = g$ a.e. $(m)$.

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3. (30p) Let $\{a_k\}$ be a sequence of complex numbers such that $\sum_{k=0}^{\infty} a_k$ is convergent. Set $S_m^n := \sum_{k=m}^{n} a_k$.