Exercise 1.4. Let $\varphi \in C_c^\infty(\mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function with compact support and $\varphi * f$ be the convolution of φ and f;

$$\varphi * f(x) := \int_{\mathbb{R}} \varphi(x-y) f(y)\, dy.$$

(a) Show $\frac{d}{dx}(\varphi * f)(x) = (\varphi' * f)(x)$ for all $x \in \mathbb{R}$.

(b) Show $(\varphi' * f)(x) = (\varphi * f')(x)$ for all $x \in \mathbb{R}$.

(c) Explain why there exists $f_n \in C_c^\infty(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^\infty(\mathbb{R},m)} = 0 = \lim_{n\to\infty} \|f' - f_n'\|_{L^1(\mathbb{R},m)}.$$

Exercise 1.5. Suppose that $\{f_n\}_{n=1}^\infty \subset L^3(\mu)$ such that $\lim_{n\to\infty} \int_X f_n \varphi d\mu$ exists in $\mathbb{C}$ for all $\varphi \in L^{3/2}(\mu)$. Show $M := \sup \|f_n\|_{L^3(\mu)} < \infty$.

Exercise 1.6. Suppose $f : X \to [-1,1]$ is a measurable function and $\varphi : [-1,1] \to \mathbb{R}$ is a bounded Borel measurable function. Show:

(a) $\|M_{\varphi\circ f}\|_{B(L^2(\mu))} \le \|\varphi\|_u := \sup_{|x|\le 1} |\varphi(x)|$ where $\|M_{\varphi\circ f}\|_{B(L^2(\mu))} = \sup_{\|h\|_{L^2(\mu)}=1} \|M_{\varphi\circ f} h\|_{L^2(\mu)}$ is the operator norm of $M_{\varphi\circ f}$.

(b) Suppose $\varphi_n : [-1,1] \to \mathbb{R}$ are bounded Borel measurable functions converging boundedly to φ, then, for all $h \in L^2(\mu)$,

$$L^2(\mu)\text{-}\lim_{n\to\infty} M_{\varphi_n\circ f} h = M_{\varphi\circ f} h.$$

(c) Show by example that it is possible that $\lim_{n\to\infty} \|M_{\varphi_n\circ f}\|_{B(L^2(\mu))} \ne 0$ even though $\varphi_n \to 0$ boundedly.

Exercise 1.7. Let $f, g : X \to [-1,1]$ be measurable functions and $U : L^2(X,\mu) \to L^2(X,\mu)$ be a unitary map such that $UM_fU^{-1} = M_g$. Let $\mathcal{H}$ denote the collection of bounded Borel measurable functions, $\varphi : [-1,1] \to \mathbb{R}$, such that $UM_{\varphi\circ f}U^{-1} = M_{\varphi\circ g}$. Show:

(a) $\varphi \in \mathcal{H}$ if $\varphi(x) = \sum_{n=0}^N a_n x^n$ is a polynomial with $a_n \in \mathbb{R}$.

(b) Show $C([-1,1], \mathbb{R}) \subset \mathcal{H}$.

(c) Show $\mathcal{H}$ contains all bounded real measurable functions.

Hints: 1. The results of Exercise 1.6 are useful. **2.** For (a) show $\varphi(M_f) = M_{\varphi\circ f}$. **3.** For (c), notice that $UM_{\varphi\circ f}U^{-1} = M_{\varphi\circ g}$ iff

$$UM_{\varphi\circ f}U^{-1}h = M_{\varphi\circ g}h \text{ for all } h \in L^2(\mu).$$

4. You do not have to prove (b) if you can prove (c).