(4) Suppose $T : L^2([0,1]) \to L^5([0,1])$ is an **everywhere** defined (but not necessarily bounded) linear operator. If there exists $C_2 < \infty$ such that $\|Tf\|_2 \le C_2 \|f\|_2$ for all $f \in L^2([0,1])$, then show $T : L^2([0,1]) \to L^5([0,1])$ is bounded, i.e. there exists $C_5 < \infty$ such that $\|Tf\|_5 \le C \|f\|_2$ for all $f \in L^2([0,1])$.

Hint: $T : L^2([0,1]) \to L^5([0,1])$ is **everywhere** defined.

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(5) For $f \in C_c((-1,1))$, let $\hat{f}(\xi)$ be the Fourier transform of f defined by

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{(-1,1)} f(x) e^{-ix\xi} dx = \int_{(-1,1)} f(x) e^{-ix\xi} \mathbf{d}x$$

and let

$$\mathcal{C} := \left\{ f \in C_c((-1,1)) : |f(0)| + \int_{\mathbb{R}} |\xi| \left|\hat{f}(\xi)\right| d\xi \le 1 \right\}.$$

(a) Show $\mathcal{C} \subset C_c^1(-1,1)$.

(b) Show $\mathcal{C}$ is a precompact subset of $C([-1,1])$.

Remark. In this problem we are identifying $f \in C_c((-1,1))$ with elements of $C([-1,1])$ by defining $f(\pm 1) = 0$. Similarly you may wish to view $f \in C_c((-1,1))$ as a function on $\mathbb{R}$ by setting $f(x) = 0$ if $|x| \ge 1$.

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(6) Let A_k be a sequence of real $n \times n$ **invertible** matrices such that $A_k \to I$ as $k \to \infty$ and $f \in L^3(\mathbb{R}^n, m)$.

(a) Compute $\|f \circ A_k\|_3$ in terms of $\|f\|_3$ and use this to show there exists a constant $C < \infty$ such that

$$\|f \circ A_k\|_3 \le C \|f\|_3 \text{ for all } f \in L^3(\mathbb{R}^n, m) \text{ and } k = 1, 2, 3, \ldots.$$

(b) Under the additional assumption that $f \in C_c(\mathbb{R}^n)$ show

$$\lim_{k \to \infty} \|f \circ A_k - f\|_3 = 0.$$

(c) Now show $\lim_{k\to\infty} \|f \circ A_k - f\|_3 = 0$ for all $f \in L^3(\mathbb{R}^n, m)$.

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(7) If μ be a complex measure on $\mathcal{B}_{[-\pi/2,\pi/2]}$ such that

$$\int_{[-\pi/2,\pi/2]} \sin^n(x) d\mu = 0 \text{ for } n = 1, 2, \ldots.$$

then $\mu = \mu([-\pi/2, \pi/2])\, \delta_0$ where δ_0 is the Dirac measure at 0, i.e. $\delta_0(A) := 1_A(0)$ for all $A \in \mathcal{B}_{[-\pi/2,\pi/2]}$. **Hint:** Consider the measure $\nu := \mu - \mu([-\pi/2, \pi/2])\, \delta_0$.