1. Analysis Qualifying Exam, Spring 2002

Notation: Throughout this test, $\mathcal{B}$ denotes the Borel σ – algebra on $\mathbb{R}$ or $\mathbb{R}^n$, $\mathcal{B}_{[a,b]}$ is the Borel σ – algebra on $[a,b]$, m denotes Lebesgue measure on $\mathcal{B}$ and $C([a,b])$ is the Banach space of continuous functions on $[a,b]$ equipped with the supremum norm. Also $L^p([a,b]) = L^p([a,b], \mathcal{B}_{[a,b]}, m)$ and $L^p(\mathbb{R}^n) = L^p(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, m)$ equipped with the usual L^p – norms.

Instructions: Clearly explain and justify your answers. You may cite theorems from textbooks or that were proved in class as long as they are not what the problem explicitly asks you to prove. Make sure to state the results that you are using and be sure to verify their hypotheses. All problems have **equal** value.

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(1) In each case below find L (allowing for values of $\pm\infty$) and justify the calculations:

$$\text{a)} \qquad L = \lim_{n\to\infty} \int_0^\pi e^{-n\sin(x)}\,dx,$$

$$\text{b)} \qquad L = \lim_{N\to\infty} \sum_{k=0}^{N} \int_0^N \frac{x^k}{k!} e^{-x}\,dx \text{ and}$$

$$\text{c)} \qquad L = \int_0^\infty \left[\int_0^\infty e^{-y/x} e^{-x^2/2} dx \right] dy$$

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(2) Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of absolutely continuous functions such that $f_n' \in L^1(\mathbb{R}, m)$ and $c := \lim_{n\to\infty} f_n(0)$ exists in $\mathbb{R}$. Further assume there exists $g \in L^1(\mathbb{R}, m)$ such that $\lim_{n\to\infty} \int_{\mathbb{R}} |g(x) - f_n'(x)| dx = 0$.

(a) Show that $f(x) \equiv \lim_{n\to\infty} f_n(x)$ exists **for all** $x \in \mathbb{R}$.

(b) Show that f is absolutely continuous and $f'(x) = g(x)$ for m- a.e. x.

Hint: As usual, integration can be used to prove theorems about differentiation.

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(3) Let H be a separable Hilbert space, $\{e_n\}_{n=1}^\infty$ be an orthonormal basis for H and $\{u_n\}_{n=1}^\infty$ be an orthonormal subset of H such that $\delta_n := e_n - u_n$ satisfies

$$\sum_{n=1}^{\infty} \|\delta_n\|^2 = \alpha < 1. \tag{1.1}$$

Show $\{u_n\}_{n=1}^\infty$ is also an orthonormal basis for H.

Hint: For $x \in H$ such that $(x, u_n) = 0$ for all n, use the given assumptions to estimate $\|x\|^2$.
