1. Analysis Qualifying Exam, Spring 2001

Notation: Throughout this test, $\mathcal{B}$ denotes the Borel σ – algebra on $\mathbb{R}$, $\mathcal{B}_{[0,1]}$ is the Borel σ – algebra on $[0,1]$, m denotes Lebesgue measure on $\mathcal{B}$ and $C([0,1])$ is the Banach space of continuous functions on $[0,1]$ equipped with the supremum norm. Also $L^p([0,1]) = L^p([0,1], \mathcal{B}_{[0,1]}, m)$ and $L^p(\mathbb{R}) = L^p(\mathbb{R}, \mathcal{B}, m)$ equipped with the usual L^p – norms.

Instructions: Clearly explain and justify your answers. You may cite theorems from textbooks or that were proved in class as long as they are not what the problem explicitly asks you to prove. Make sure to state the results that you are using and be sure to verify their hypotheses.

Complete problem (1) and 6 of the remaining 7 problems. Please tell me on the front of the test which problem I should **not** grade.

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(1) For each of the following, determine if the statement is true (always) or false (not always true). If true, give a brief explanation; false, give a counter example.

   (a) If $f, g \in C([0,1])$ and $f(x) = g(x)$ for m – a.e. x then $f(x) = g(x)$ for all $x \in [0,1]$.

   (b) To every function $f \in L^1([0,1])$ there is a continuous function $g : [0,1] \to \mathbb{C}$ such that $f(x) = g(x)$ for m – a.e. x.

   (c) If μ and ν are positive measures on $\mathcal{B}_{[0,1]}$ such that $\mu([0,x]) = \nu([0,x])$ for all $x \in [0,1]$ then $\mu = \nu$.

   (d) If μ and ν are positive finite measures on $\mathcal{B}_{[0,1]}$ such that $\mu([0,x]) = \nu([0,x])$ for all $x \in [0,1]$ then $\mu = \nu$.

Do 6 of the following problems.

(2) Suppose that $G : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function and for $f \in L^2([0,1])$ and $x \in [0,1]$ let

$$Tf(x) = \int_0^1 G(x,y) f(y) dm(y). \tag{1.1}$$

   (a) Show $Tf \in C([0,1])$ and that $T : L^2([0,1]) \to C([0,1])$ is a bounded operator.

   (b) Show that T takes bounded subsets of $L^2([0,1])$ to precompact subsets of $C([0,1])$.

(3) Let $f \in L^2([0,1])$, $g(x) := Tf(x)$ be as in Eq. (1.1) with $G(x,y) := \min(x,y)$. Show $g \in C^1((0,1))$ and $g'(x) = \int_x^1 f(y) dm(y)$.

Hint: Either (a) compute the derivative using the definition of the derivative or (b) prove the result first for "nice" f and then pass to the limit.

(4) Let X and Y be topological spaces, $A \subset X$ and $B \subset Y$ be subsets. Show $\overline{A \times B} = \bar{A} \times \bar{B}$. (If you get stuck in the general case, prove the result when X and Y are metric spaces to get partial credit.)