

4. Suppose $T_n : X \rightarrow Y$ for $n \in \mathbb{N}$ is a sequence of bounded linear operators between two Banach spaces, X and Y . Further assume that $\lim_{n \rightarrow \infty} T_n x$ exists for all $x \in X$. Show $Tx := \lim_{n \rightarrow \infty} T_n x$ defines a **bounded** linear operator from X to Y .

5. Suppose that $\{\nu_n\}_{n=1}^{\infty}$ are complex measures and μ is a finite positive measure on a measurable space, (X, \mathcal{M}) . Further let $|\nu_n|$ denotes the total variation measure associated to ν_n .

(a) If $\sum_{n=1}^{\infty} |\nu_n|(X) < \infty$, then $\nu := \sum_{n=1}^{\infty} \nu_n$ is a complex measure.

(b) If $|\nu_n(A)| \leq \mu(A)$ and $\nu(A) := \lim_{n \rightarrow \infty} \nu_n(A)$ exists for all $A \in \mathcal{M}$, then $\nu : \mathcal{M} \rightarrow \mathbb{C}$ is a complex measure.

6. Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be a sequence of absolutely continuous functions such that $f'_n \in L^1(\mathbb{R}, m)$ and $c := \lim_{n \rightarrow \infty} f_n(0)$ exists in \mathbb{R} . Further assume there exists $g \in L^1(\mathbb{R}, m)$ such that $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |g(x) - f'_n(x)| dx = 0$.

(a) Show that $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ exists for all $x \in \mathbb{R}$.

(b) Show that f is absolutely continuous and $f'(x) = g(x)$ for m - a.e. x .

Hint: As usual, integration can be used to prove theorems about differentiation.

7. Let $h \in C([-1, 1], \mathbb{R})$ be a one to one function and let Z denote the space of functions of the form $p(h) := \sum_{k=1}^n a_k h^k$ for some $n \in \mathbb{N}$ and $a_k \in \mathbb{R}$. Show Z is a dense subspace of $L^1([-1, 1], \mathcal{B}_{[-1, 1]}, m)$. **Note:** if necessary, for partial credit, you may further assume that h is never zero.

8. Let $G : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded Borel measurable function. Define $f_0(t) = 1$ and $f_n : [-1, 1] \rightarrow \mathbb{R}$ inductively by

$$f_{n+1}(t) = 1 + \int_0^t G(f_n(\tau)) d\tau.$$

Show:

(a) f_n are well defined and that $f_n \in C([-1, 1], \mathbb{R})$ for all n .

(b) The sequence $\{f_n\}_{n=1}^{\infty}$ has a uniformly convergent subsequence.