Qualifying Exam in Real Analysis, September 9, 2003

Instructions: No books or notes may be used in this exam. Do any **four** of the following five problems, **including number 1**. You may cite without proof any result in the text by Folland. Please indicate which three of problems 2 to 5 that you would like to have graded.

1. (90 pts.) For each of the following, determine if the statement is true (always) or false (not always true). If true, give a brief proof; if false, give a counterexample or prove it is false in some other way. No credit for just deciding if it is true or false. An answer like "This is a theorem in Folland" without either naming the theorem or giving further details is not acceptable. Unless otherwise indicated, assume that $(X, \mathcal{M}, \mu)$ is a measure space.

(a) If $E_1 \supset E_2 \supset \dots$ is a sequence of subsets of X in $\mathcal{M}$ such that $\cap E_n = \emptyset$, then $\lim_{n\to\infty} \mu(E_n) = 0$.

(b) If $\mu(X) < \infty$, and $\{f_n\}$ is a sequence of real-valued measurable functions on X satisfying

$$C \geq f_1(x) \geq f_2(x) \dots \quad \forall x \in X,$$

where C is a real constant, then

$$\lim_{n\to\infty} \int_X f_n(x)dx = \int_X \lim_{n\to\infty} f_n(x)dx.$$

(c) If $\{x_n\}$ is a sequence in a Banach space X such that $\{f(x_n)\}$ is a Cauchy sequence for every $f \in X^*$, the dual of X, then $\{x_n\}$ is a Cauchy sequence in X.

(d) If $\{a_{ij}\}_{1\leq i,j<\infty}$ is a doubly indexed sequence in $\mathbb{R}$ with $\sum_{i,j} |a_{ij}| < \infty$, then

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}\right) = \lim_{m\to\infty}\left(\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}\right).$$

(e) The Fourier transform of any function in $L^2(\mathbb{R})$ is continuous.

(f) If T is any nonzero distribution on $\mathbb{R}$, then the distribution xT is again nonzero.

2. (30 pts.) Prove the Monotone Convergence Theorem as a consequence of Fatou's Lemma.