Do either problem 7 or problem 8 below. If you work both problems 7 and 8, please be clear which problem you want me to grade. I will **only** grade one of these problems.

---

(7) Let $H = L^2([0,1], m)$, $k \in L^2([0,1] \times [0,1], m \otimes m)$ and for $f \in H$ let

$$Kf(x) := \int_{[0,1]} k(x,y) f(y) dy.$$

Show:

(a) $Kf(x)$ is well defined for m – a.e. x, i.e. $y \to k(x,y)f(y)$ is integrable for m – a.e. x.

(b) $\|K\|_{op} \le \|k\|_{L^2(m \otimes m)}$.

(c) $\|K\|_{HS} = \|k\|_{L^2(m \otimes m)}$.

See Problem 5 for the definitions of $\|K\|_{op}$ and $\|K\|_{HS}$.

---

(8) (a) Let $g \in L^\infty(\mathbb{R}, m)$, $K := \|g\|_\infty < \infty$ and $f(x) := \int_0^x g(y) dy$. Then

(1.3) $$|f(y) - f(x)| \le K |x - y| \text{ for all } x, y \in \mathbb{R}.$$

(b) Conversely suppose $f : \mathbb{R} \to \mathbb{R}$ is any function such that $f(0) = 0$ and f satisfies Eq. (1.3). Show for m – a.e. x that $f'(x)$ exists, $|f'(x)| \le K$ and also show

$$f(x) = \int_0^x f'(y) dy \text{ for } \textbf{all } x \in \mathbb{R}.$$