

- (4) Suppose that  $A, B : H \rightarrow H$  are everywhere defined (but not necessarily bounded) linear operators. Show that if  $A$  and  $B$  satisfy the algebraic relation,

$$(Ax, y) = (x, By) \text{ for all } x, y \in H,$$

then  $A$  and  $B$  are necessarily bounded. **Hint:**  $A, B : H \rightarrow H$  are everywhere defined.

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- (5) Let  $H$  be a separable Hilbert space,  $K : H \rightarrow H$  be a bounded linear operator,  $\{e_n\}_{n=1}^{\infty}$  and  $\{u_m\}_{m=1}^{\infty}$  be an orthonormal bases for  $H$ . Show:

(a)  $\sum_{n=1}^{\infty} \|Ke_n\|^2 = \sum_{m=1}^{\infty} \|K^*u_m\|^2$  allowing for the possibility that the sums are infinite.

(b) Use part (a) to conclude  $\|K\|_{HS}^2 := \sum_{n=1}^{\infty} \|Ke_n\|^2$  is independent of the choice of orthonormal basis  $\{e_n\}_{n=1}^{\infty}$  and use this to show

$$\|K\|_{HS} \geq \|K\|_{op} := \sup \{ \|Kh\| : h \in H \ni \|h\| = 1 \}.$$

(c) Assume  $\|K\|_{HS}^2 < \infty$  and let  $K_N x := \sum_{n=1}^N (Kx, e_n)e_n$  for all  $x \in H$ . Show  $\|K - K_N\|_{op} \rightarrow 0$  as  $N \rightarrow \infty$ .

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- (6) Let  $X$  be a Banach space and  $B := \{x \in X : \|x\| < 1\}$ .

(a) Suppose  $G$  and  $\{G_n\}_{n=1}^{\infty}$  are subsets of  $X$  such that  $\bar{G}_n$  is compact for each  $n$  and

$$\delta_n := \sup \{ \text{distance}(x, G_n) : x \in G \} \downarrow 0 \text{ as } n \rightarrow \infty$$

Conclude  $\bar{G}$  is also a compact subset of  $X$ . (**Hint:** Show  $G$  is totally bounded.)

(b) An operator  $K : X \rightarrow X$  is said to be compact if  $G := K(B)$  is precompact in  $X$ . If  $K_n : X \rightarrow X$  are compact operators and  $K : X \rightarrow X$  is a bounded operator such that  $\|K - K_n\|_{op} \rightarrow 0$  as  $n \rightarrow \infty$ , then  $K$  is a compact operator.

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