1. ANALYSIS QUALIFYING EXAM, FALL 2002

Instructions: Clearly explain and justify your answers. You may cite theorems from textbooks or that were proved in class as long as they are not what the problem explicitly asks you to prove. Make sure to state the results that you are using and be sure to verify their hypotheses. All problems have **equal** value and you should do problems 1 – 6 **and either** problem 7 or problem 8. If you work both problems 7 and 8, please be clear which problem you want me to grade. I will **only** grade one of these problems.

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(1) Let $\gamma(t) := e^{i2\pi t} = \cos 2\pi t + i \sin 2\pi t$ and for $x \in \mathbb{R} \setminus \{\pm 1\}$ let

$$N(x) := \frac{1}{2\pi i} \int_0^1 \frac{\dot{\gamma}(t)}{\gamma(t) - x} dt. \tag{1.1}$$

(a) Show that it is permissible to differentiate under the integral in Eq. (1.1) to conclude $N'(x)$ exists and

$$N'(x) = \frac{1}{2\pi i} \int_0^1 \frac{\dot{\gamma}(t)}{(\gamma(t) - x)^2} dt \text{ for all } x \in \mathbb{R} \setminus \{\pm 1\}. \tag{1.2}$$

(b) Conclude from Eq. (1.2) that $N'(x) = 0$ for all $x \in \mathbb{R} \setminus \{\pm 1\}$. (**Hint:** compute $\frac{d}{dt}(\gamma(t) - x)^{-1}$.)

(c) Find $N(x)$ for all $x \in \mathbb{R} \setminus \{\pm 1\}$. (**Hint:** compute $N(0)$ and "$N(\pm\infty)$.")

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(2) Suppose that μ is a complex measure on the Borel σ – algebra on $[0,1] \times [0,\infty)$ such that

$$\int_{(x,y) \in [0,1] \times [0,\infty)} x^m e^{-ny} d\mu(x,y) = 0 \text{ for all } m = 0,1,2,\ldots \text{ and } n = 1,2,\ldots.$$

Show $\mu \equiv 0$.

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(3) Let μ and ν be two finite positive measures on the Borel σ – algebra on $\mathbb{R}^n$ which satisfy

$$\int_{\mathbb{R}^n} f d\mu \le \int_{\mathbb{R}^n} f d\nu \text{ for all } f \in C_c(\mathbb{R}^n, [0,\infty)).$$

Show:

(a) $\mu(K) \le \nu(K)$ for all compact subsets $K \subset \mathbb{R}^n$.

(b) $\mu(B) \le \nu(B)$ for all Borel sets $B \subset \mathbb{R}^n$.

(c) There exists a Borel measurable function $g : \mathbb{R}^n \to [0,1]$ such that

$$\int_{\mathbb{R}^n} f d\mu = \int_{\mathbb{R}^n} f g d\nu$$

holds for all bounded measurable functions $f : \mathbb{R}^n \to \mathbb{C}$.