# Qualifying Exam in Real Analysis, September 12, 2000

**Instructions**: No books or notes may be used in this exam. Do all seven problems, **including number 1**. You may cite without proof any theorem proved in the text by Folland (Chapters 1–6 only), but you must reprove any result given in the exercises.

1. For each of the following, determine if the statement is true (always) or false (not always true). If true, give a brief proof; if false, give a counterexample. In the following $(X, \mathcal{M}, \mu)$ will denote an arbitrary measure space.

(a) If $f \in L^1(X)$ with $f(x) \geq 0$ a.e. and $\int_X f(x)d\mu = 0$, then $f = 0$ a.e.

(b) If $f_n, f \in L^1(X)$ and $f_n \to f$ in $L^1$, then $f_n \to f$ a.e.

(c) If $f_n, f \in L^1(X)$ and $f_n \to f$ in $L^1$, then $f_n \to f$ in measure.

(d) Let $f \in L^1(\mathbb{R}, dm)$, where $dm$ denotes Lebesgue measure and put $F(x) = \int_0^x f(t)dt$. Then $F$ is almost everywhere differentiable and $F'(x) = f(x)$.

(e) Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces such that as sets $\mathcal{Y} \subset \mathcal{X}$. (It is not assumed that the norm on $\mathcal{Y}$ is induced by that of $\mathcal{X}$.) Then every bounded linear functional on $\mathcal{Y}$ extends to a (not necessarily unique) bounded linear functional on $\mathcal{X}$.

2. Let $K$ be a nonempty, convex, closed subset of a Hilbert space $\mathcal{H}$. Show that there is a unique vector in $K$ with minimal norm.

3. Prove the following generalized dominated convergence theorem. Let $X$ be a measure space and let $f_n, g_n, f, g \in L^1(X)$, $n \geq 1$. Assume that the following hold

(i) $f_n \to f$, $g_n \to g$ a.e.
(ii) $|f_n| \leq g_n$, $n \geq 1$,
(iii) $\int g_n \to \int g$.

Show that $\int f_n \to \int f$.

4. For $n \geq 1$, consider the sequence $f_n \in C^\infty(\mathbb{R})$ defined by

$$f_n(x) = \frac{n}{nx + i},$$

Show that the sequence $\{f_n\}$ is convergent in $\mathcal{D}'(\mathbb{R})$ and find its limit.

5. Let $X$ be a closed subspace of $L^p([0,1])$, $1 \leq p < \infty$, and assume that $X \subset C([0,1])$. Show that there exists a positive constant $C$ such that for all $f \in X$ the following holds:

$$\sup_{x \in [0,1]} |f(x)| \leq C\|f\|_p.$$

6. Let $X$ and $Y$ be compact Hausdorff spaces. Show that the set of all functions of the form $f(x,y) = \sum_{i=1}^k g_i(x)h_i(y)$, $k = 1, 2, \dots$ where $g_i \in C(X)$ and $h_i \in C(Y)$ is dense in $C(X \times Y)$.

7. If $f \in L^p \cap L^\infty$, show that

$$\|f\|_\infty = \lim_{q \to \infty} \|f\|_q$$