1. Linear algebra

Question 1.1.

(a) Consider λ_i, $\lambda_j \in \text{eig}(A)$ such that $\lambda_i \neq \lambda_j$. Let (x_i, y_i) and (x_j, y_j) denote the right and left eigenvectors of A associated with λ_i and λ_j. Show that $y_i^* x_j = 0$.

(b) Let x denote an eigenvector of A associated with an eigenvalue λ. Prove that if λ has a left-eigenvector y such that $y^* x = 0$, then $\text{am}(\lambda) > 1$.

Question 1.2. Given $A \in M_{m,n}$ with $m \geq n$, prove that there exists a unique $U \in M_{m,n}$ with orthonormal columns, and a unique Hermitian positive semidefinite $H \in M_n$ such that $A = UH$.

2. Group Theory

Question 2.1. Let p be a prime number.

(a) Show that the order of $\widehat{1+p}$ in $(\mathbb{Z}/p^2\mathbb{Z})^\times$ is equal to p.

(b) Use **(a)** above to construct a non-abelian group of order p^3.

(c) Describe the non-abelian group you have constructed in **(b)** above via generators and relations.

Note. As usual, $(\mathbb{Z}/p^2\mathbb{Z})^\times$ denotes the multiplicative group consisting of all the congruence classes $\widehat{x} \in \mathbb{Z}/p^2\mathbb{Z}$, such that $\gcd(x, p) = 1$.

Question 2.2. Let G be a group. Let $r \geq 2$ be an integer. Assume that G contains a non–trivial subgroup H of index $[G : H] = r$. Prove the following.

(a) If G is simple, then G is finite and $|G|$ divides $r!$.

(b) If $r \in \{2, 3, 4\}$, then G cannot be simple.

(c) For all integers $r \geq 5$, there exist simple groups G which contain non-trivial subgroups H of index $[G : H] = r$.