(6) (30 pts.) Let $\langle A, +, \cdot \rangle$ be a commutative ring with identity 1 and let $<$ be a linear order on A such that for all a, b, x in A

(I) $a < b \Rightarrow a + x < b + x$ and

(II) $a < b, 0 < x \Rightarrow a \cdot x < b \cdot x$.

(a) Prove that $\langle A, +, \cdot \rangle$ is an integral domain.

(b) Let $A^+ = \{a \in A : 0 < a\}$. Prove the following:

(i) A^+ is closed under multiplication and addition.

(ii) If $a \in A$, then exactly one of the following holds: $a \in A^+, -a \in A^+, a = 0$.

(iii) $1 \in A^+$.

(7) (40 pts.)

Consider the equations

$$\begin{aligned} x^2 + 3y^2 &= 4 \\ x^2 - xy + 2y^2 &= 4 \end{aligned}$$

(a) Let I be the ideal of $\mathbf{C}[x, y]$ generated by these equations. Find the Groebner basis for I relative to lexicographic order where $x > y$.

(b) Find a Groebner basis for $\mathbf{C}[x] \cap I$.

(c) Find all solutions to these equations that lie $\mathbf{C}^2$.

(d) Find a vector space basis for $\mathbf{C}[x, y]/I$.

(8) (40 pts.) Consider the matrix

$$A = \begin{pmatrix} i & -0 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that A generates a cyclic group of G of order 4.

(b) Find the Hilbert series of $\mathbf{C}[x, y]^G$.

(c) Show that $\mathbf{C}([x, y]^G$ is Cohen-Macaulay by explictly finding generators and separators for $\mathbf{C}[x, y]^G$.