

Question 1. (10 points) Let $A \in M_n$ be triangular with $\text{am}(\lambda) = \text{gm}(\lambda) = 1$ for all $\lambda \in \sigma(A)$.

- (a) Prove that the matrices of left- and right-eigenvectors of A are triangular.
- (b) Given the A of part (a), let D be a nondefective matrix such that

$$D = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where $C \in M_2$ and $\text{am}(\lambda) = 2$ for $\lambda \in \sigma(C)$. Find a matrix X such that $D = X\Lambda X^{-1}$ with Λ diagonal. (You may assume that you know the triangular matrices from part (a).)

Question 2. (10 points) Consider the Sylvester operator $S(X) = AX - XB$ for $A \in M_m$, $B \in M_n$ and $X \in M_{m,n}$.

- (a) Prove that S is a linear transformation.
- (b) Derive the matrix representation of S in terms of the standard basis for $M_{2,1}$.
- (c) Prove that the Sylvester operator $S(X) = AX - XB$ is nonsingular if and only if $\sigma(A) \cap \sigma(B) = \emptyset$.

Question 3. (20 points) For positive integers s, t, j, ℓ with $\text{gcd}(s, \ell) = 1$, let $G_{s,t,j,\ell}$ be the group with generators a, b and relations $a^s = e$, $b^t = a^j$, $bab^{-1} = a^\ell$. A group H is said to be *metacyclic* if H has a normal subgroup N such that N is cyclic and H/N is also cyclic.

- (a) Prove that if H is a finite metacyclic group, then there is a homomorphism from some $G_{s,t,j,\ell}$ onto H .
- (b) Let $G = G_{s,t,j,\ell}$. Prove that G is metacyclic and that $|G| \mid st$.
- (c) Give an example of s, t, j, ℓ with $\text{gcd}(s, \ell) = 1$ such that $|G_{s,t,j,\ell}| < st$.