

**Question 3.2.** Let  $R$  be a commutative ring and let  $M$  and  $N$  be  $R$ -modules. Prove the following.

- (a) If  $M$  and  $N$  are projective  $R$ -modules, then  $M \otimes_R N$  is a projective  $R$ -module.
- (b) If  $M$  and  $N$  are flat  $R$ -modules, then  $M \otimes_R N$  is a flat  $R$ -module.
- (c) If  $M$  is a flat  $R$ -module and  $N$  is an injective  $R$ -module, then  $\text{Hom}_R(M, N)$  is an injective  $R$ -module.
- (d) Let  $p$  be a prime number and let  $\mathbb{Z}_{(p)}$  denote the localization of  $\mathbb{Z}$  with respect to its prime ideal  $p\mathbb{Z}$ . Show that  $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_{(p)}, \mathbb{Q}/\mathbb{Z})$  is an injective  $\mathbb{Z}$ -module.

**Question 3.3.** Let  $n$  be a squarefree integer greater than 3. Let  $R$  denote the subring  $\mathbb{Z}[\sqrt{-n}] = \{a + b\sqrt{-n} \mid a, b \in \mathbb{Z}\}$  of the field of complex numbers  $\mathbb{C}$ .

- (a) Show that  $\sqrt{-n}$  and  $1 + \sqrt{-n}$  are irreducible in  $R$ .
- (b) Prove that  $R$  is not a unique factorization domain (UFD).
- (c) Construct an ideal in  $R$  which is not principal.

**Hint.** In a UFD, an element is irreducible if and only if it is prime.

#### 4. Field Theory and Galois Theory

**Question 4.1.**

- (a) If  $\mathbb{F}_q$  is a finite field with  $q$  elements, show that  $\mathbb{F}_q^\times$  is a cyclic group.
- (b) Show that for each integer  $n \geq 1$ , there exists an irreducible polynomial over  $\mathbb{F}_q$  of degree  $n$ .
- (c) Consider the map  $\phi : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^n}$  given by  $\phi(x) = x^q$ . Note that  $\phi$  is an  $\mathbb{F}_q$ -linear endomorphism of the  $\mathbb{F}_q$ -vector space  $\mathbb{F}_{q^n}$ . Find the characteristic and minimal polynomials of  $\phi$ .

**Question 4.2.**

- (a) Let  $p$  be an odd prime. Show that  $\sin(\frac{2\pi}{p})$  is algebraic over  $\mathbb{Q}$  and determine its degree over  $\mathbb{Q}$ .
- (b) Show that  $\mathbb{Q}(\sin(\frac{2\pi}{p}))$  is a Galois extension of  $\mathbb{Q}$  and determine its Galois group over  $\mathbb{Q}$ .
- (c) Find all  $p$ 's such that  $p^{1/3} \in \mathbb{R}$  is contained in  $\mathbb{Q}(\sin(\frac{2\pi}{p}))$ .