Question 1. Let $T : M_n(\mathbb{R}) \mapsto M_n(\mathbb{R})$ be the transformation such that

$$T(X) = \frac{1}{2}(X - X^T).$$

(a) Prove that T is a linear transformation.

(b) Determine the null space of T and find its dimension.

(c) Derive the matrix representation of T in terms of the standard basis for M_3.

Question 2. Prove that a triangular matrix is normal if and only if it is diagonal.

Question 3. Assume that (λ, x) is an eigenpair of $A \in \mathbb{C}^{n\times n}$ such that $\text{am}(\lambda) = \text{gm}(\lambda) = 1$. Prove that there exists a nonsingular matrix $(\, x \quad X \,)$ with inverse $(\, y \quad Y \,)^*$ such that

$$\begin{pmatrix} y^* \\ Y^* \end{pmatrix} A (\, x \quad X \,) = \begin{pmatrix} \lambda & 0 \\ 0 & M \end{pmatrix}.$$

Question 4. Let G be a finite abelian group of order n. Suppose that G has a unique subgroup of order d for each positive divisor of n. Prove that G is cyclic.

Question 5. Prove that a group of order 120 is not simple.

Question 6. Let G be a group whose center has index n. Show that every conjugacy class in G has at most n elements.