

**Question 1.**

- (a) State and prove the Schur decomposition theorem.
- (b) Let  $A \in M_n$  with  $\lambda_i \in \sigma(A)$ . Prove that if  $\text{trace}(A^*A) = \sum_{i,j=1}^n |a_{ij}|^2 = \sum_{i=1}^n |\lambda_i|^2$ , then  $A$  is unitarily diagonalizable.

**Question 2.** Consider  $A \in M_n$ ,  $B \in M_m$  and  $X \in M_{n,m}$ . Prove that the Sylvester operator  $S(X) = AX - XB$  is nonsingular if and only if  $\sigma(A) \cap \sigma(B) = \emptyset$ .

**Question 3.** Define the dual norm  $\varphi(x)$  of a vector norm  $\|x\|$  on  $\mathbb{C}^n$ . Given symmetric positive-definite  $B \in M_n(\mathbb{R})$ , find the dual norm of the elliptic norm  $\|x\|_B$ .

**Question 4.** Let  $G$  be a group of order  $p^n$ ,  $p$  a prime. Show that any nontrivial normal subgroup of  $G$  intersects the center of  $G$  nontrivially.

**Question 5.** How many elements of order 7 are there in a simple group of order 168?

**Question 6.** Let  $H, K$  be two subgroups of finite index in the group  $G$ . Show that  $H \cap K$  is of finite index, too.