5.(30 points) Let K be a degree n extension of the finite field $\mathbb{Z}_p$ where p is a prime.

a. Prove that K is the splitting field of the polynomial $X^{p^n} - X$.

b. Prove that $X^{p^n} - X$ is the product of all the irreducible polynomials in $\mathbb{Z}_p[X]$ of degrees which divide n.

c. Use (b) to determine how many monic irreducible degree three polynomials there are in $\mathbb{Z}_2[X]$.

6.(25 points) Determine the splitting field and the Galois group of the polynomial $X^4 - 3$.

7.(25 points) Consider the algebraic variety $V = Z((X^3 - Y^2))$. Prove that V is not isomorphic to the affine space $\mathbb{A}^1$. Hint: Suppose they are isomorphic, consider integral closures and then derive a contradiction.

8.(25 points) (a) State Gauss' Lemma.

(b) Suppose that f and g are polynomials with rational coefficients and the product fg has integer coefficients. Prove that the product of any coefficient of f with any coefficient of g is an integer.

9.(25 points) Let R be a Noetherian ring and let N be the set of all nilpotent elements in R.

(a) Prove that N is the finite intersection of all the minimal prime ideals in R.

(b) Let $P_1, \cdots, P_t$ be the associated prime ideals for a minimal primary decomposition of the zero ideal. Prove that the union of these prime ideals is the full set of zero divisors in R.

10.(25 points)

(a) State the Noether Normalization Lemma.

(b) Give an example of how it is used.

11.(20 points) Prove that the nonzero elements in a finite field form a cyclic group.