May 31, 2000

# QUALIFYING EXAM
# ALGEBRA

1.(25 points) Let $G$ be a finite group and $p$ a prime dividing the order of $G$.

(a) Prove the 3rd Sylow Theorem, that the number of $p$-Sylow subgroups is congruent to one mod $p$ and divides the order opf $G$.

(b) Suppose that $|G| = 7 \cdot 11 \cdot 13 = 1001$. Classify all possible such $G$.

2.(35 points) Let $G$ be the finite group $Gl(2, \mathbb{F}_5)$ of invertible $2 \times 2$ matrices with entries in the finite field $\mathbb{F}_5$

(a) Show that the order of this group is $480 = 2^5 \cdot 3 \cdot 5$.

(b) Prove that the number of 5-Sylow subgroups in $G$ is 6. Hint: The matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with $a \in \mathbb{F}_5$ form a 5-Sylow subgroup.

(c) Prove that every matrix in any of these 5-Sylow subgroups has characteristic polynomial $p(X) = (X-1)^2$.

(d) Why is it possible that the degree two polynomial $p(X)$ has more than two matrix solutions to the equation $p(X) = 0$?

3.(40 points) (a) State and prove the singular-value decomposition theorem for $A \in M_{m,n}$ with rank $r$.

(b) If $A$ has singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Show that

$$\sigma_i = \max_{\substack{S \\ \dim(S)=i}} \min_{\substack{x \in S \\ x \neq 0}} \frac{\|Ax\|_2}{\|x\|_2}.$$

4.( 20 points) (a) Prove that the group of group automorphisms $Aut(\mathbb{Z}_5 \oplus \mathbb{Z}_5)$ is isomorphic to the group $Gl(2, \mathbb{F}_5)$ in 2 above.

(b) Prove that there exists a noncommutative semidirect product of $\mathbb{Z}_5$ and $Z_5 \oplus \mathbb{Z}_5$ .