Question 3.2. Consider the $\mathbb{Z}$–algebra $R := \mathbb{Z}[X]/(2X+1)$.

(a) Show that there is a ring isomorphism $R \simeq S^{-1}\mathbb{Z}$, where S is the multiplicatively closed subset of $\mathbb{Z}$ consisting of all the non-negative integral powers of 2, i.e. $S := \{1, 2, 2^2, 2^3, \dots\}$.

(b) Is R a flat $\mathbb{Z}$–module? Justify your answer.

(c) Is R a projective $\mathbb{Z}$–module? Justify your answer.

(d) Is R an injective $\mathbb{Z}$–module? Justify your answer.

Question 3.3. Let $R := \mathbb{Z}[\sqrt{10}] = \{a + b\sqrt{10} \mid a, b \in \mathbb{Z}\}$, viewed as a subring of the field of complex numbers with the usual operations.

(a) Show that the group of units $R^\times$ of R is infinite.

(b) Show that the element 2 is irreducible in R.

(c) Show that the element 2 is not prime in R.

(d) Show that every element in R can be written as a product of irreducible elements.

(e) Is R a unique factorization domain? Justify your answer.

Hint. Recall that there is a multiplicative norm map $N : \mathbb{Z}[\sqrt{10}] \longrightarrow \mathbb{Z}$, given by $N(a + b\sqrt{10}) = a^2 - 10b^2$, for all $a, b \in \mathbb{Z}$.

4. Field Theory and Galois Theory

Question 4.1. Find the splitting fields and Galois groups of the following polynomials over $\mathbb{Q}$. You should state clearly results that you use.

(a) $(x^2+3)(x^3-5)$.

(b) $(x^2-3)(x^3-7)$.

(c) $x^{15}-2$.

(d) x^3+2x^2+1.