1. Linear Algebra

Question 1.1. Assume that (λ, x) is an eigenpair of $A \in M_n$ such that $\mathrm{am}(\lambda) = \mathrm{gm}(\lambda) = 1$. Prove that there exists a nonsingular matrix $\begin{pmatrix} x & X \end{pmatrix}$ with inverse $\begin{pmatrix} y & Y \end{pmatrix}^*$ such that

$$\begin{pmatrix} y^* \\ Y^* \end{pmatrix} A \begin{pmatrix} x & X \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & M \end{pmatrix}.$$

Question 1.2. Given $A \in M_{m,n}$ with $m \geq n$, prove that there exists a unique $U \in M_{m,n}$ with orthonormal columns, and a unique Hermitian positive semidefinite $H \in M_n$ such that $A = UH$.

(*State in detail any auxiliary results that you use.*)

2. Group Theory

Question 2.1. Let G be a group and let $\mathcal{Z}(G)$ denote its center.

(a) Show that if $G/\mathcal{Z}(G)$ is cyclic then $G = \mathcal{Z}(G)$.

(b) Show that if $\mathrm{card}(G) = p^3$, for some prime number p and G is non-commutative then $\mathrm{card}(\mathcal{Z}(G)) = p$.

(c) Construct a non-commutative group G of cardinality (order) 16 whose center $\mathcal{Z}(G)$ is not cyclic.

Note. As usual, $\mathrm{card}(X)$ denotes the cardinality of the set X.

Question 2.2. Let n be an integer. Let G_n be the group given by generators and relations as follows.

$$G_n = \langle x, y \mid x^3 = 1,\ xyx^{-1} = y^n \rangle$$

(a) Provide (with proof) necessary and sufficient conditions on the integer n for the group G_n to be **finite.**

(b) Assuming that G_n is finite, compute its order as a function of n.

3. Ring Theory and Module Theory

Question 3.1. Let X and Y be two independent variables.

(a) Prove that the ring $R_1 := \mathbb{Q}(X)[Y]/((Y^2 + X)^3)$ is a local ring.

(b) Prove that the ring $R_2 := \mathbb{Q}[X, Y]/((Y^2 + X)^3)$ is not a local ring by giving examples (with proof) of two distinct maximal ideals in R_2.

(c) Give an example of a prime ideal in R_2 which is not maximal. Are there such ideals in R_1? Justify your answers.

Note. As usual, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}(X)$ denotes the field of rational functions of variable X with coefficients in $\mathbb{Q}$, $\mathbb{Q}(X)[Y]$ denotes the ring of polynomials of variable Y with coefficients in $\mathbb{Q}(X)$, and $\mathbb{Q}[X, Y]$ denotes the ring of polynomials of variables X, Y with coefficients in $\mathbb{Q}$. Also, recall that a commutative ring is called **local** if it has a unique maximal ideal.