# Algebra Qualifying Examination, September, 2001 Part 3

1. Let $F$ be a field. Which of the following pairs of rings are isomorphic?
   a) $XF[X]$ and $X^2F[X]$.
   b) $XF[X]$ and $(X-1)F[X]$.

2. Determine the splitting field $F$ of $X^6 - 9$ over $\mathbb{Q}$. What is its dimension over $\mathbb{Q}$ and describe its Galois group, $Gal(F/\mathbb{Q})$?

3. Let $F$ be an algebraically closed field of characteristic $p > 0$. Show that for each $n$ there is exactly one subfield of $F$ whose cardinality is $p^n$. Show that the union of those fields is the algebraic closure of the subfield of cardinality $p$.

4. Find the Jordan canonical form over the complex numbers of the matrix

$$\begin{bmatrix} 2 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 1 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

5. Let $F$ be an algebraically closed field. We say that two matrices $A, B \in M_n(F)$ ($n \times n$ matrices) are conjugate(or similar) if there exists an invertible matrix $g \in M_n(F)$ such that $gAg^{-1} = B$. Let $f(X) \in F[X]$ be a monic polynomial of degree $n$. Let $S_f$ be the set of all $A \in M_n(F)$ with characteristic polynomial $f$.
   a) Show that being conjugate is an equivalence relation. We will call the set of all $A \in M_n(F)$ conjugate to $B$ the conjugacy class of $B$.
   b) Show that if $A$ and $B$ are conjugate then and if $A \in S_f$ then $B$ is also in $S_f$.
   c) Show that $S_f$ is a finite union of conjugacy classes. (Hint: Look at the Jordan cnaonical form).