Chapter 2. Derivatives

2.1. The Derivative as a Function

Definition. Derivative Function.

The *derivative* of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Note. Motivated by Section 1.5, we see that $f'(x)$ is the slope of the line tangent to $y = f(x)$ as a function of x.

Note. There are a number of ways to denote the derivative of $y = f(x)$:

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}[f].$$

Example. Page 157 number 3.

Note. We can also calculate higher *order* derivatives:

$$y'' = \frac{d}{dx}[y'], y''' = \frac{d}{dx}[y''], y^{(4)} = \frac{d}{dx}[y'''], \ldots, y^{(n)} = \frac{d}{dx}[y^{(n-1)}].$$

Example. Page 157 number 10.

Rule 1. Derivative of a Constant Function.

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}[c] = 0.$$

Proof. From the definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to o} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$

QED

Rule 2. Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

Note. Before we present the proof of the Power Rule, we introduce the Binomial Theorem.

Theorem. Binomial Theorem

Let a and b be real numbers and let n be a positive integer. Then

$$\begin{aligned}(a+b)^n &= a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \ldots + nab^{n-1} + b^n \\ &= \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} a^{n-i}b^i\end{aligned}$$

where $\begin{pmatrix} n \\ i \end{pmatrix} = \frac{n!}{(n-i)!i!}$ and $i! = (i)(i-1)(i-2)\cdots(3)(2)(1)$.

Note. We can prove the Binomial Theorem using *Mathematical Induction.*

Proof of the Power Rule. By definition,

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{\Sigma_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i}h^i - x^n}{h} \\ &= \lim_{h\to 0} \frac{x^n + \Sigma_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i}h^i - x^n}{h}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{\Sigma_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i} h^{i}}{h} \\
&= \lim_{h\to 0} \frac{h\,\Sigma_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i} h^{i-1}}{h} \\
&= \lim_{h\to 0} \sum_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i} h^{i-1} \\
&= \lim_{h\to 0} nx^{n-1} + \sum_{i=2}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i} h^{i-1} \\
&= nx^{n-1}
\end{aligned}
$$

QED

Note. See page 150 for a proof of the Power Rule that doesn't (explicitly) use the Binomial Theorem. Can you find the **error** in the computation (it's subtle and conceptual, but does not affect the conclusion)?

Rule 3. Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}[cu] = c\frac{du}{dx}.$$

Rule 4. Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}[u + v] = \frac{du}{dx} + \frac{dv}{dx}.$$

Note. The proofs of Rules 3 and 4 follow from the corresponding rules for limits (namely, the Constant Multiple Rule and the Sum Rule, respectively).

Corollary. If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, then $P'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-1} + \cdots + 2a_2x + a_1$.

Example. Page 157 number 12.

Theorem 1. Differentiability Implies Continuity

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof. By definition, we need to show that $\lim_{x \to c} f(x) = f(c)$, or equivalently that $\lim_{h \to 0} f(c+h) = f(c)$. Then

$$\begin{aligned} \lim_{h \to 0} f(c+h) &= \lim_{h \to 0} \left(f(c) + \frac{f(c+h) - f(c)}{h} \cdot h \right) \\ &= \lim_{h \to 0} f(c) + \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \cdot \lim_{h \to 0} h \\ &= f(c) + f'(c) \cdot 0 \\ &= f(c). \end{aligned}$$

Therefore f is continuous at $x = c$. *QED*

Theorem 2. Intermediate Value Property of Derivatives

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

Example. Page 159 number 34.